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ASYMPTOTIC BEHAVIOUR OF WIENER–HOPF FACTORS OF A RANDOM WALK

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For a random walk governed by a general distribution function F on $(-\infty, +\infty)$, we establish the exponential and subexponential asymptotic behaviour of the corresponding right Wiener–Hopf factor F^+. The results apply to classes of distribution functions in recent publications: the subexponential class $\mathscr{S}$ and a related (exponential) class $\mathscr{S}_\gamma$. Given the behaviour of F^+, the Wiener–Hopf identity is used, to obtain the behaviour of F. To reverse the argument, we derive a new identity, similar in form to the first one. The results for F^+ are then fruitfully applied to give a full description of the tail behaviour of the maximum of the random walk. Also they provide new proofs for recent theorems on the tail of the waiting-time distribution in the GI/G/1 queue.

Random walk	Wiener–Hopf factorization
maximum-distribution	subexponential
GI/G/1 queue.	distribution functions

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, identically distributed with common (non-degenerate) distribution function $F(x)$ on $(-\infty, \infty)$. Let $S_0 = 0$, $S_n = X_1 + \ldots + X_n$ $(n \geq 1)$, then the sequence $\{S_n\}$ constitutes the random walk generated by F.

Let $N = \min\{n > 0: S_n > 0\}$, $\bar{N} = \min\{n > 0: S_n \leq 0\}$ be the first ascending and first descending ladder-epochs of $\{S_n\}$.

For the characteristic function

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

the following Wiener–Hopf factorization is available [8, p. 570]:

$$1 - f(t) = [1 - f_+(t)] \cdot [1 - f_-(t)] \tag{1}$$

where f_+ and f_- are the Fourier-Stieltjes transforms of the (possibly defective) random variables S_N and $S_{\bar{N}}$.

For $1-f_+(t)$ and $1-f_-(t)$ we have the following representations [8, pp. 569, 570]:

$$1-f_+(t)=\exp\left[-\sum_{n=1}^{\infty} n^{-1}\int_{0^+}^{\infty} e^{itx}\,dF^{(n)}(x)\right] \quad (\operatorname{Im} t \geq 0),$$

$$1-f_-(t)=\exp\left[-\sum_{n=1}^{\infty} n^{-1}\int_{-\infty}^{0^+} e^{itx}\,dF^{(n)}(x)\right] \quad (\operatorname{Im} t \leq 0),$$

where $F^{(n)}$ denotes the n-fold convolution of F.

Writing F_+ and F_- for the (possibly defective) distribution functions of S_N and $S_{\bar{N}}$, we obtain from (1) the following identity

$$F=F_++F_--F_+*F_- \tag{2}$$

representing F in terms of distribution functions on $(-\infty, 0]$ and $(0,\infty)$.

Put

$$A=\sum_{n=1}^{\infty} n^{-1}\mathbf{P}[S_n\leq 0], \qquad B=\sum_{n=1}^{\infty} n^{-1}\mathbf{P}(S_n>0).$$

Then, since $A+B=\infty$, only three cases can occur:

(i) $A<\infty$ (and $B=\infty$); i.e. $F_+(\infty)=1$ and $F_-(0^+)=1-e^{-A}$
(ii) $B<\infty$ (and $A=\infty$); i.e. $F_+(\infty)=1-e^{-B}$ and $F_-(0^+)=1$
(iii) $A=B=\infty$; i.e. $F_+(\infty)=F_-(0^+)=1$.

Call $\mu=\int_{-\infty}^{\infty} x\,dF(x)$ (where μ is defined iff $\int_{-\infty}^{\infty}|x|\,dF(x)<\infty$), then it is well-known that the cases (i), (ii) and (iii) are implied by $\mu>0$, $\mu<0$, $\mu=0$ respectively. The converse implications are not true.

2. Tails of distribution functions

Let G be a distribution function on $[0,\infty)$ with Laplace–Stieltjes transform

$$\tilde{g}(\lambda)=\int_0^{\infty} e^{-\lambda x}\,dG(x).$$

Definition 1. G belongs to the class $\mathcal{S}$ of subexponential distribution functions if

$$\lim_{x\to\infty}\frac{1-G^{(2)}(x)}{1-G(x)}=2.$$

This class has been introduced by Chistyakov [3]. Further references are Athreya–Ney [1] and Teugels [15, 16]. If $G\in\mathcal{S}$, then the right tail of G decreases slower than any exponential, i.e.

$$e^{\rho x}[1-G(x)]\to\infty \quad \text{for all } \rho>0 \text{ as } x\to\infty.$$

An extensively studied subclass of $\mathcal{S}$ is the class of distribution functions with regularly varying tails, characterized by: $1-G(x)\sim x^{-\alpha}L(x)$ where $\alpha\geqslant 0$ and L is slowly varying at infinity; see e.g. [8].

Definition 2. For $\gamma>0$, we say that G belongs to the class $\mathcal{S}_\gamma$ if $\tilde{g}(-\gamma)<\infty$ and

$$G_\gamma(x)\equiv\frac{1}{\tilde{g}(-\gamma)}\int_0^x e^{\gamma y}\,dG(y)\in\mathcal{S}.$$

It follows from the proof of [16, Theorem 1] that for $G\in\mathcal{S}_\gamma$:

(i) $\lim_{x\to\infty}(1-G^{(2)}(x))/(1-G(x))=\beta>2$,

(ii) $\lim_{x\to\infty}(1-G(x-b))/(1-G(x))=e^{\gamma b}$, uniformly for b in compact sets,

(iii) $\tilde{g}(-\gamma)=\beta/2<\infty$.

From these facts, we obtain the important conclusion that our Definition 2 is equivalent to the original definition of Chover–Ney–Wainger [4, 5], who introduced classes of distribution functions, satisfying (i)–(iii). Indeed, their class $\mathcal{S}(d)$ coincides with $\mathcal{S}_\gamma$ for $d=\beta/2$. Also $\mathcal{S}_0\equiv\mathcal{S}$. If $G\in\mathcal{S}_\gamma$ $(\gamma>0)$, then the right tail of G decreases more rapidly than that of an exponential with parameter γ.

An important tool in the proof of our results will be the following closure property: if $G\in\mathcal{S}_\gamma$ and $1-G(x)\sim c[1-H(x)]$ $(c>0)$, then $H\in\mathcal{S}_\gamma$. The proof can be found in [16, Corollary 5] and relies on the following expression:

$$1-G_\gamma(x)=\frac{1}{\tilde{g}(-\gamma)}\left[e^{\gamma x}(1-G(x))+\gamma\int_x^\infty e^{\gamma y}(1-G(y))\,dy\right].$$

3. Tail behaviour of the right Wiener–Hopf factor

Introduce for $\lambda\geqslant 0$:

$$f_+(i\lambda)=\int_{0^+}^\infty e^{-\lambda x}\,dF_+(x)\equiv g_+(\lambda)$$

$$f_-(-i\lambda)=\int_{-\infty}^{0^+} e^{\lambda x}\,dF_-(x)\equiv g_-(\lambda).$$

It is well-known that

$$f(i\lambda)=\int_{-\infty}^\infty e^{-\lambda x}\,dF(x)$$

converges in a strip $-\alpha_f<\operatorname{Re}\lambda<\beta_f(\alpha_f,\beta_f\geqslant 0)$ and that $g_+(\lambda)$ converges in the half plane $-\alpha_+<\operatorname{Re}\lambda$ $(\alpha_+\geqslant 0)$. It follows from the identity (1) that $\alpha_f=\alpha_+$. From now on, we will denote this common value by γ.

If $\gamma>0$, we can use the classical theorems for Laplace–Stieltjes transforms [18] in order to obtain first information on the tail behaviour, namely: if $1-F(x)$ is exponentially bounded (i.e. $1-F(x)\leqslant Ke^{-\alpha x}$ for some K and $\alpha>0$), then $F_+(x)$ is exponentially bounded and conversely. It is shown in Theorem 1 (A) how this statement can be made much more explicit if more is known about the nature of the singularity in $-\gamma$.

If $\gamma = 0$, the question arises how to find non-exponential estimates for the right tail of F_+ in terms of similar behaviour of F. This is carried out in Theorem 1 (B, C).

From now on, we use the following notation:

$$F_0(x) = F(x) \cdot U_0(x)$$

where $U_0(x)$ is a distribution function concentrated at 0.

$$F_1(x) = m^{-1} \int_0^x [1 - F(t)]\,dt$$

where $m = \int_0^\infty [1 - F(t)]\,dt$.

$$F^c(x) = F(\infty) - F(x).$$

Also, if G is a defective distribution function, then $G \in \mathcal{S}_\gamma$ stands for

$$G(x)/G(\infty) \in \mathcal{S}_\gamma.$$

Theorem 1. *Let $-\gamma$ be the left abscissa of convergence of $f(i\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x}\,dF(x)$.*

(A) *If $\gamma > 0$, then*

$$F_0 \in \mathcal{S}_\gamma \tag{3}$$

iff

$$F_+ \in \mathcal{S}_\gamma \tag{4}$$

and each of the above implies

$$\lim_{x\to\infty} \frac{F^c(x)}{F_+^c(x)} = 1 - g_-(\gamma). \tag{5}$$

(B) *If $\gamma = 0$ and $A < \infty$, then*

$$F_0 \in \mathcal{S} \tag{6}$$

iff

$$F_+ \in \mathcal{S} \tag{7}$$

and each of the above implies

$$\lim_{x\to\infty} \frac{F^c(x)}{F_+^c(x)} = e^{-A}. \tag{8}$$

(C) *If $\gamma = 0$, $A = \infty$ and*

$$-\infty < E(S_{\bar N}) < 0, \tag{9}$$

then

$$F_1 \in \mathcal{S} \tag{10}$$

iff

$$F_+ \in \mathscr{S} \tag{11}$$

and each of the above implies

$$\lim_{x\to\infty} \frac{\int_x^\infty [1-F(t)]\,\mathrm{d}t}{F_+^c(x)} = \int_{-\infty}^{0^+} |t|\,\mathrm{d}F_-(t). \tag{12}$$

Remark on condition (9). Rewriting (1) as

$$\frac{1-f(t)}{\mathrm{i}t} = [1-f_+(t)]\cdot\frac{1-f_-(t)}{\mathrm{i}t} \tag{13}$$

shows that for $A=\infty$ and $B<\infty$, (9) is equivalent to $\mu<0$, and that for $A=B=\infty$, (9) implies that $\mu=0$.

Also, if $B<\infty$:

$$\int_{-\infty}^{0^+} |t|\,\mathrm{d}F_-(t) = (-\mu)\mathrm{e}^B.$$

Proof of Theorem 1. (A) We first remark that (1) implies the finiteness of $f(-\mathrm{i}\gamma)$ if and only if $g_+(-\gamma)$ is finite.

From (2) we obtain that for all positive x:

$$\frac{1-F(x)}{F_+(\infty)-F_+(x)} = 1 - \int_{-\infty}^{0^+} \frac{F_+(\infty)-F_+(x-y)}{F_+(\infty)-F_+(x)}\,\mathrm{d}F_-(y). \tag{14}$$

If $F_+ \in \mathscr{S}_\gamma$, we find by dominated convergence that

$$\lim_{x\to\infty} \frac{1-F(x)}{F_+(\infty)-F_+(x)} = 1 - \int_{-\infty}^{0^+} \mathrm{e}^{\gamma y}\,\mathrm{d}F_-(y) = 1-g_-(\gamma)$$

and by the closure property of the class $\mathscr{S}_\gamma$ [16, Corollary 5] that $F_0 \in \mathscr{S}_\gamma$. Hence: (4) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5).

In order to prove that (3) $\Longrightarrow$ (4) and that (3) $\Longrightarrow$ (5) we introduce

$$F_\gamma(x) = \frac{1}{f(-\mathrm{i}\gamma)} \int_{-\infty}^{x} \mathrm{e}^{\gamma t}\,\mathrm{d}F(t)$$

$$F_{+\gamma}(x) = \frac{1}{g_+(-\gamma)} \int_{0^+}^{x} \mathrm{e}^{\gamma t}\,\mathrm{d}F_+(t).$$

We will derive an equation (see (18) below) which is similar to (14) but where the integrand involves F_γ rather than $F_{+\gamma}$. From (2) we obtain after some easy calculations:

$$f(-\mathrm{i}\gamma)F_\gamma(x) = g_-(\gamma) - g_+(-\gamma)\int_{-\infty}^{0^+} \mathrm{e}^{\gamma y}F_{+\gamma}(x-y)\,\mathrm{d}F_-(y) + g_+(-\gamma)F_{+\gamma}(x) \quad \text{if } x>0, \tag{15}$$

$$f(-i\gamma)F_\gamma(x)=\int_{-\infty}^{x} e^{\gamma y}\,dF_-(y)-g_+(-\gamma)\int_{-\infty}^{0+} e^{\gamma y}F_{+\gamma}(x-y)\,dF_-(y)\quad \text{if } x\leq 0. \tag{16}$$

If we put

$$f_\gamma(t)=\int_{-\infty}^{\infty} e^{itx}\,dF_\gamma(x),$$

$$f_{+\gamma}(t)=\int_{0}^{\infty} e^{itx}\,dF_{+\gamma}(x),$$

$$f_{-\gamma}(t)=\int_{-\infty}^{0} e^{itx}\,e^{\gamma x}\,dF_-(x),$$

we find from (15) and (16)

$$f(-i\gamma)f_\gamma(t)=g_+(-\gamma)f_{+\gamma}(t)-g_+(-\gamma)f_{+\gamma}(t)f_{-\gamma}(t)+f_{-\gamma}(t)$$

or since $f_{-\gamma}(0)=g_-(\gamma)$:

$$[1-g_-(\gamma)]f_{+\gamma}(t)=\frac{f(-i\gamma)}{g_+(-\gamma)}f_\gamma(t)g(t)-\frac{1}{g_+(-\gamma)}f_{-\gamma}(t)g(t) \tag{17}$$

where $g(t)=(1-f_{-\gamma}(0))/(1-f_{-\gamma}(t))$ is the transform of a non-defective distribution function G, concentrated on $(-\infty,0]$, namely:

$$G(x)=[1-f_{-\gamma}(0)]\sum_{n=0}^{\infty} B^{(n)}(x),$$

where $B(dx)=e^{\gamma x}\,dF_-(x)$. Inverting (17) gives, for $x>0$,

$$[1-g_-(\gamma)]F_{+\gamma}(x)=\frac{f(-i\gamma)}{g_+(-\gamma)}\int_{-\infty}^{0+} F_\gamma(x-y)\,dG(y)-\frac{g_-(\gamma)}{g_+(\gamma)}$$

and since $f(-i\gamma)=g_+(-\gamma)+g_-(\gamma)-g_+(-\gamma)g_-(\gamma)$, we obtain

$$1-F_{+\gamma}(x)=\frac{f(-i\gamma)}{[1-g_-(\gamma)]g_+(-\gamma)}\int_{-\infty}^{0+}[1-F_\gamma(x-y)]\,dG(y). \tag{18}$$

If (3) is valid, we have that

$$\lim_{x\to\infty}\frac{1-F_{+\gamma}(x)}{1-F_\gamma(x)}=\frac{f(-i\gamma)}{[1-g_-(\gamma)]g_+(-\gamma)}$$

and hence $F_+\in\mathcal{S}_\gamma$. So (3) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (5).

(B) The proof given in (A) is also valid if $\gamma=0$ provided $g(0)=1-e^{-A}<1$, i.e. if $A<\infty$.

(C) If $A=\infty$, then F_- is non-defective and (2) gives for all $t>0$:

$$1-F(t)=\int_{-\infty}^{0+}[F_+(t-y)-F_+(t)]\,dF_-(y).$$

Integrating from $x>0$ to b $(b>x)$ we find

$$\int_x^b [1-F(t)]\,\mathrm{d}t = \int_{-\infty}^{0+} \mathrm{d}F_-(y) \int_x^b [F_+(t-y)-F_+(t)]\,\mathrm{d}t. \tag{19}$$

Since

$$\int_x^b [F_+(t-y)-F_+(t)]\,\mathrm{d}t = \int_x^{x-y} [F_+(\infty)-F_+(t)]\,\mathrm{d}t - \int_b^{b-y} [F_+(\infty)-F_+(t)]\,\mathrm{d}t$$

we find, for $y \leq 0$:

$$-y[F_+(b)-F_+(x-y)] \leq \int_x^b [F_+(t-y)-F_+(t)]\,\mathrm{d}t \leq -y[F_+(\infty)-F_+(x)].$$

Substituting this in (19), letting b tend to ∞ and using condition (9), we obtain that

$$m_F(x) \equiv \int_x^\infty [1-F(t)]\,\mathrm{d}t$$

satisfies

$$\int_{-\infty}^{0+} |y|[F_+(\infty)-F_+(x-y)]\,\mathrm{d}F_-(y) \leq m_F(x) \leq [F_+(\infty)-F_+(x)] \int_{-\infty}^{0+} |y|\,\mathrm{d}F_-(y). \tag{20}$$

Also, for $s>0$,

$$1 \leq \int_{-\infty}^{0+} |y|\,\mathrm{d}F_-(y) \cdot \frac{F_+(\infty)-F_+(x+s)}{m_F(x+s)} \leq \frac{\int_{-\infty}^{0+} |y|\,\mathrm{d}F_-(y)}{\int_{-s}^{0+} |y|\,\mathrm{d}F_-(y)} \cdot \frac{m_F(x)}{m_F(x+s)}. \tag{21}$$

From (20) and (21) we get that, for $a>0$,

$$\lim_{x\to\infty} \frac{m_F(x+a)}{m_F(x)} = 1 \iff \lim_{x\to\infty} \frac{F_+(\infty)-F_+(x+a)}{F_+(\infty)-F_+(x)} = 1.$$

Since $m_F(x) = m[1-F_1(x)]$ we have that (10) implies

$$\lim_{x\to\infty} \frac{m_F(x+a)}{m_F(x)} = 1 \text{ or equivalently that } \lim_{x\to\infty} \frac{F_+(\infty)-F_+(x+a)}{F_+(\infty)-F_+(x)} = 1.$$

From (20) this gives

$$\lim_{x\to\infty} \frac{\int_x^\infty [1-F(t)]\,\mathrm{d}t}{F_+(\infty)-F_+(x)} = \int_{-\infty}^{0+} |y|\,\mathrm{d}F_-(y).$$

Hence: (10) $\Longrightarrow$ (12) and from the closure property of the class $\mathcal{S}$: (10) $\Longrightarrow$ (11). Conversely, (11) implies that

$$\lim_{x\to\infty} \frac{F_+(\infty)-F_+(x+a)}{F_+(\infty)-F_+(x)} = 1$$

and so (12) and (10) are satisfied.

4. Maximum of a random walk

In this section we apply the above results to find the asymptotic behaviour of the maximum-distribution of the random walk $\{S_n\}$, in terms of the corresponding tail behaviour of the governing distribution function F. Let for $x \geq 0$:

$$W(x) = \mathbf{P}\left[\sup_{n \geq 0} S_n \leq x\right],$$

then it is well-known [8, 13] that W is a proper distribution function on $[0, \infty)$ if and only if $B < \infty$ and then the Laplace-Stieltjes transform $w(\lambda)$ of W is given by

$$w(\lambda) = \exp\left[-\sum_{n=1}^{\infty} n^{-1} \int_{0^+}^{\infty} (1 - e^{-\lambda x}) \mathrm{d}F^{(n)}(x)\right]. \tag{22}$$

As before, let $-\gamma \leq 0$ be the left abscissa of convergence of $f(\mathrm{i}\lambda)$.

The case $f(-\mathrm{i}\gamma) \geq 1$ is well-known. Indeed, in this case there exists a unique positive real number κ such that $f(-\mathrm{i}\kappa) = 1$, and the famous Cramer-estimates are available. (See Theorem 2(A) (i), (ii). For proofs we refer to [8, 9, 17].)

In the case $\gamma > 0$, but $f(-\mathrm{i}\gamma) < 1$, we still have that the right tail of F decreases exponentially fast and a natural class to consider is the class $\mathscr{S}_\gamma$ (see Theorem 2(A) (iii)).

If $\gamma = 0$, the tail of F decreases more slowly than any exponential, and here we will consider the class $\mathscr{S}$ (Theorem 2(B)).

Theorem 2. *Suppose $B < \infty$. Let $-\gamma$ be the left abscissa of convergence of $f(\mathrm{i}\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x} \mathrm{d}F(x)$.*

(A) *Suppose $\gamma > 0$, then*

(i) *if $f(-\mathrm{i}\gamma) > 1$:*

$$1 - W(x) \sim K e^{-\kappa x} \quad (x \to \infty)$$

where

$$f(-\mathrm{i}\kappa) = 1 \quad (0 < \kappa < \gamma),$$

$$K = \frac{e^{-B}}{\kappa} \cdot \frac{1}{-g_+'(-\kappa)} = \frac{e^{-B}}{\kappa} \cdot \frac{1 - g_-(\kappa)}{-f'(-\mathrm{i}\kappa)}.$$

(ii) *If $f(-\mathrm{i}\gamma) = 1$ and $|f'(-\gamma)| < \infty$ we have the same result as in* (i) *with $\kappa = \gamma$. If $f(-\mathrm{i}\gamma) = 1$ and $|f'(-\mathrm{i}\gamma)| = \infty$, then*

$$1 - W(x) = o(e^{-\gamma x}) \quad (x \to \infty).$$

(iii) *If $f(-\mathrm{i}\gamma) < 1$,*

$$1 - W(x) = o(e^{-\gamma x}) \quad (x \to \infty),$$

but

$$F_0 \in \mathscr{S}_\gamma \tag{23}$$

$$\Leftrightarrow F_+ \in \mathcal{S}_\gamma \tag{24}$$

$$\Leftrightarrow W \in \mathcal{S}_\gamma \tag{25}$$

and each of the above implies

$$\lim_{x\to\infty} \frac{1-W(x)}{1-F(x)} = \frac{e^{-B}}{[1-g_+(-\gamma)][1-f(-i\gamma)]}\,. \tag{26}$$

(B) *Suppose* $\gamma = 0$; *then if* $\mu < 0$,

$$F_1 \in \mathcal{S} \tag{27}$$

$$\Leftrightarrow F_+ \in \mathcal{S} \tag{28}$$

$$\Leftrightarrow W \in \mathcal{S} \tag{29}$$

and each of the above implies

$$\lim_{x\to\infty} \frac{1-W(x)}{\int_x^\infty [1-F(t)]\,dt} = -\frac{1}{\mu}\,.$$

Proof. (A) To prove (iii) we rewrite (22) as follows:

$$w(\lambda) = \frac{e^{-B}}{1-g_+(\lambda)}\,.$$

Inverting the Laplace–Stieltjes transform gives

$$W(x) = e^{-B} \sum_{n=0}^{\infty} F_+^{(n)}(x).$$

Since $f(-i\gamma) < 1$ we have that $g_+(-\gamma) < 1$ and hence, applying [16, Theorem 5] we obtain that

$$(24) \Leftrightarrow (25) \Leftrightarrow \lim_{x\to\infty} \frac{1-W(x)}{F_+(\infty)-F_+(x)} = \frac{e^{-B}}{[1-g_+(-\gamma)]^2}\,.$$

Combining the above relations with our Theorem 1(A) gives the desired result.

(B) Applying again the result of [16] we obtain that

$$(28) \Leftrightarrow (29) \Leftrightarrow \lim_{x} \frac{1-W(x)}{F_+(\infty)-F_+(x)} = e^{B}.$$

The result now follows from Theorem 1(C).

5. Waiting-time in a GI/G/1 queue

The results of Theorem 2 are immediately applicable to the limiting behaviour of the waiting-time distribution in a GI/G/1 queueing system. Indeed, by Lindley's theorem [6], the stationary waiting-time distribution $W(x)$ is the same as the distribution of $\sup_{n\geq 0} S_n$, where $\{S_n\}$ is the random walk generated by the random variables $X_n = U_n - T_n$ where U_n is the service-time of the nth customer and T_n is the interarrival-time between the nth and $(n+1)$th customer. The $\{U_n\}_1^\infty$ are non-negative, independent and identically distributed random variables with distribution function $B(x)$ and expectation β. Similarly the $\{T_n\}_1^\infty$ are non-negative, independent and identically distributed with distribution function $A(x)$ and expectation α. Moreover $\{U_n\}$ and $\{T_n\}$ are mutually independent.

Let F denote the distribution function of X_n; then $f(i\lambda) = b(\lambda) \cdot a(-\lambda)$ where b and a are the Laplaces–Stieltjes transforms of B and A. It follows that the left abscissa of convergence of $f(i\lambda)$ coincides with that of $b(\lambda)$.

Since

$$F(x) = \int_0^\infty B(x+y)\,dA(y)$$

it follows easily that (with the notation above):

$$F_0 \in \mathcal{S}_\gamma \iff B \in \mathcal{S}_\gamma \text{ and } \lim_{x\to\infty} \frac{1-F(x)}{1-B(x)} = a(\gamma) \quad \text{if } \gamma > 0,$$

$$F_1 \in \mathcal{S} \iff B_1 \in \mathcal{S} \text{ and } \lim_{x\to\infty} \frac{\int_x^\infty [1-F(t)]\,dt}{\int_x^\infty [1-B(t)]\,dt} = 1 \quad \text{if } \gamma = 0.$$

Together with Theorem 2 this gives the following corollary.

Corollary. *Let $\beta < \alpha$. Let $-\gamma$ be the left abscissa of convergence of $b(\lambda) = \int_0^\infty e^{-\lambda x}\,dB(x)$.*

(A) *If $\gamma > 0$ and $b(-\gamma)\cdot a(\gamma) < 1$,*

$$B \in \mathcal{S}_\gamma \iff W \in \mathcal{S}_\gamma$$

and each implies

$$\lim_{x\to\infty} \frac{1-W(x)}{1-B(x)} = \frac{w(-\gamma)a(\gamma)}{1-b(-\gamma)a(\gamma)}.$$

(B) *If $\gamma = 0$:*

$$B_1 \in \mathcal{S} \iff W \in \mathcal{S}$$

and each implies

$$\lim_{x\to\infty} \frac{1-W(x)}{\int_x^\infty [1-B(t)]\,dt} = \frac{1}{\alpha - \beta}.$$

6. Bibliographical comments

Most of the results presented in this paper first appeared in the author's doctoral thesis [17] and for a special case in [14]. We want to stress the generality of our method of examining tail behaviour of ladder-height and maximum-distribution of a general random walk. Our results for the classes $\mathcal{S}_\gamma$ ($\gamma \geq 0$) give a unified treatment of several partial results in literature, obtained for related classes of tail behaviour and mostly in the context of queueing theory.

The result of Corollary (B) for the GI/G/1 queue has been proved (using different methods) by Pakes [10, Theorem 1], generalizing a similar result of Cohen [7], obtained for regularly varying tail behaviour.

The result of Corollary (A) is also partially contained in Pakes [10, Lemmas 2 and 4], who considered the class $\mathcal{S}(d)$, mentioned in Section 2.

We also have to refer to the papers of Rogozin [11], Borovkov [2] and Smith [12] who obtained fragments of our Theorems 1 and 2 for different classes of tail behaviour (functions of moderate growth, sub-power functions, ...). We remark that our method unifies this kind of theorems in their strongest from.

Finally we mention that in [17], similar results are proved for the tail behaviour of the supremum-distribution of certain processes with stationary independent increments.

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